Fixed Points of a Random Permutation

Reproduction and Revised Editions

Source. This document is an independent LaTeX reproduction of, and editorial adaptation based on, the mathematical paper shown in the Bilibili video *【日常研究】摆烂小能手掀起的数学问题* (video ID `BV1Fh411N71S`):

`https://www.bilibili.com/video/BV1Fh411N71S`

Repository. The LaTeX sources, compiled PDFs, and edition notes are published at:

`https://github.com/Ritsuka314/opuscula`

PDF editions.

`paper-en.pdf` `paper-en-revised.pdf` `paper-zh.pdf`

Purpose. The reconstruction was prepared solely for entertainment and educational purposes. It is unofficial, is not presented as an author-approved transcript, and makes no claim of affiliation with the video creator or platform.

The three editions.

1. **Reproduced English edition.** A close typesetting of the four-page English note visible in the video. It intentionally retains the source's nonstandard wording, omissions, notation, and induction argument.
2. **Revised English edition.** A corrected, self-contained mathematical report. It defines the model precisely, proves the derangement formula by inclusion–exclusion, gives a unified probability distribution, and proves the expectation with indicator variables.
3. **Revised Chinese edition.** A polished Chinese counterpart to the revised English report. It follows the same definitions, results, and proofs rather than translating the reproduced wording literally.

Edition in this file: Revised English edition
This edition is a corrected, self-contained report based on the mathematical question in the source video.

Typeset with LaTeX from the source video cited above.

Fixed Points of a Random Permutation

Distribution and Expected Value

Abstract

Let a permutation of n distinct objects be chosen uniformly at random, and let X denote its number of fixed points. This report derives the exact distribution of X by counting derangements and proves that $\mathbb{E}[X] = 1$ for every positive integer n. The result models, for example, the number of correct answers obtained when n distinct choices are assigned uniformly at random to n distinct blanks.

1 Problem setup

For a positive integer n, write

$$[n] = \{1, 2, \ldots, n\}.$$

Let $\sigma : [n] \to [n]$ be a permutation chosen uniformly from the $n!$ elements of the symmetric group S_n. An index $r \in [n]$ is a *fixed point* of σ if $\sigma(r) = r$. Define

$$X = \big|\{r \in [n] : \sigma(r) = r\}\big|.$$

Our goals are to determine the probability mass function of X and its expected value.

2 Counting derangements

A *derangement* is a permutation with no fixed points. Let D_m denote the number of derangements of m objects, with the convention $D_0 = 1$.

Theorem 1 (Derangement formula)**.** *For every integer $m \geq 0$,*

$$D_m = m! \sum_{j=0}^{m} \frac{(-1)^j}{j!}.$$

Consequently, a uniformly random permutation of m objects is a derangement with probability

$$\frac{D_m}{m!} = \sum_{j=0}^{m} \frac{(-1)^j}{j!}.$$

Proof. For $r \in [m]$, let A_r be the set of permutations that fix r. If a set $J \subseteq [m]$ has j elements, then exactly $(m-j)!$ permutations fix every element of J. Inclusion–exclusion therefore gives

$$\begin{aligned} D_m &= \left| \bigcap_{r=1}^{m} \overline{A_r} \right| \\ &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)! \\ &= m! \sum_{j=0}^{m} \frac{(-1)^j}{j!}. \end{aligned}$$

Dividing by the total number $m!$ of permutations proves the probability formula. $\square$

When $m \geq 2$, the terms for $j=0$ and $j=1$ cancel. Thus the same probability may also be written as

$$\sum_{j=2}^{m} \frac{(-1)^j}{j!},$$

which is the form used in the reproduced note.

3 Distribution of the number of fixed points

Theorem 2. *For $k = 0, 1, \ldots, n$,*

$$\Pr(X = k) = \frac{\binom{n}{k} D_{n-k}}{n!} = \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}.$$

Proof. First choose the k indices that will be fixed; this can be done in $\binom{n}{k}$ ways. The remaining $n-k$ indices must then be deranged, which can be done in D_{n-k} ways. Hence exactly $\binom{n}{k} D_{n-k}$ permutations have k fixed points. Dividing by $n!$ and substituting the derangement formula yields

$$\begin{aligned} \Pr(X = k) &= \frac{\binom{n}{k} D_{n-k}}{n!} \\ &= \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}. \end{aligned}$$

$\square$

The unified formula includes the two boundary cases:

$$\Pr(X = n) = \frac{1}{n!}, \qquad \Pr(X = n-1) = 0.$$

The second identity holds because a permutation cannot move exactly one element while fixing every other element.

4 Expected number of fixed points

Theorem 3. *For every $n \geq 1$,*

$$\mathbb{E}[X] = 1.$$

Proof. For each $r \in [n]$, define the indicator variable

$$I_r = \begin{cases} 1, & \sigma(r) = r, \\ 0, & \sigma(r) \neq r. \end{cases}$$

Then $X = \sum_{r=1}^{n} I_r$. Of the $n!$ permutations, exactly $(n-1)!$ fix a specified index r, so

$$\Pr(\sigma(r) = r) = \frac{(n-1)!}{n!} = \frac{1}{n} \quad \text{and} \quad \mathbb{E}[I_r] = \frac{1}{n}.$$

By linearity of expectation,

$$\mathbb{E}[X] = \sum_{r=1}^{n} \mathbb{E}[I_r] = n \cdot \frac{1}{n} = 1.$$

The indicator variables need not be independent for this argument. □

5 Conclusion

The number of fixed points in a uniformly random permutation has the exact distribution

$$\Pr(X = k) = \frac{\binom{n}{k} D_{n-k}}{n!},$$

and its mean is always one. Therefore, when n distinct answers are placed uniformly at random into n distinct blanks, one answer is correct on average. This is an expectation statement: an individual attempt may have no correct answers, one correct answer, or several.