Fixed Points of a Random Permutation

Reproduction and Revised Editions

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Publication page. The canonical page for this writing, including all three editions and their download links, is:

`https://opuscula.ritsuka.moe/euler-derangement-cloze/`

PDF editions.

`paper-en.pdf`   `paper-en-revised.pdf`   `paper-zh.pdf`

Source repository. The LaTeX sources, compiled PDFs, and edition notes are available at `https://github.com/Ritsuka314/opuscula`.

Purpose. The reconstruction was prepared solely for entertainment and educational purposes. It is unofficial, is not presented as an author-approved transcript, and makes no claim of affiliation with the video creator or platform.

The three editions.

1. **Reproduced English edition.** A close typesetting of the four-page English note visible in the video. It intentionally retains the source's nonstandard wording, omissions, notation, and induction argument.
2. **Revised English edition.** A corrected, self-contained mathematical report. It defines the model precisely, proves the derangement formula by inclusion–exclusion, gives a unified probability distribution, and proves the expectation with indicator variables.
3. **Revised Chinese edition.** A polished Chinese counterpart to the revised English report. It follows the same definitions, results, and proofs rather than translating the reproduced wording literally.

Edition in this file: Reproduced English edition
This edition preserves the text and calculation visible in the source video as closely as the available frames permit.

Typeset with LaTeX from the source video cited above.

Lemma 1. *Consider a permutation of n elements:*

$$f : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\},$$

where f is bijective. Then the probability of getting the permutation such that

$$f(k) \neq k\big(\forall k \in \{1, 2, \dots, n\}\big)$$

is

$$\sum_{i=2}^{n} \frac{(-1)^i}{i!}.$$

And the total number of such f is

$$n! \sum_{i=2}^{n} \frac{(-1)^i}{i!}.$$

The proof of this lemma is omitted here. Assume X is a random variable referring to how many elements in the permutation are corrected aligned (i.e. $X =$ the number of k's such that $f(k) = k$). Our problem is to calculate the probability distribution of X and its expectation.

The trivial cases are

$$P(X = n) = \frac{1}{n!},$$

and

$$P(X = n-1) = 0.$$

In general, for $k \in \{0, 1, 2, \ldots, n-2\}$,

$$\begin{aligned} P(X = k) &= P\big(k \text{ elements aligned with the other } (n-k) \text{ elements ALL misaligned}\big) \\ &= \frac{1}{n!}\binom{n}{k}\left(\sum_{i=2}^{n-k}\frac{(-1)^i}{i!}\right)(n-k)! \\ &= \frac{1}{k!}\sum_{i=2}^{n-k}\frac{(-1)^i}{i!}. \end{aligned}$$

where $n!$ is the total number of cases considered, $\binom{n}{k}$ is the binomial coefficient C_n^k, and

$$\left(\sum_{i=2}^{n-k}\frac{(-1)^i}{i!}\right)(n-k)!$$

is the number of cases where all $(n-k)$ fixed elements are misaligned in the alignment.

Therefore, the probability distribution of X is

$$P(X = k) = \begin{cases} \dfrac{1}{k!}\displaystyle\sum_{i=2}^{n-k}\frac{(-1)^i}{i!}, & \text{if } k \in \{0, 1, 2, \ldots, n-2\}, \\ 0, & \text{if } k = n-1, \\ \dfrac{1}{n!}, & \text{if } k = n. \end{cases}$$

Proposition 1. *The expectation of X is*

$$E(X) = 1.$$

Proof.

$$\begin{aligned} E(X) &= \frac{n}{n!} + (n-1)\cdot 0 + \sum_{k=0}^{n-2} \frac{k}{k!} \sum_{i=2}^{n-k} \frac{(-1)^i}{i!} \\ &= \frac{1}{(n-1)!} + \sum_{k=1}^{n-2} \frac{1}{(k-1)!} \sum_{i=2}^{n-k} \frac{(-1)^i}{i!} \\ &= \frac{1}{(n-1)!} + \sum_{k=1}^{n-2} \sum_{i=2}^{n-k} \frac{(-1)^i}{i!(k-1)!}. \end{aligned}$$

Use induction on n. Suppose we already have

$$\frac{1}{(n-1)!} + \sum_{k=1}^{n-2} \sum_{i=2}^{n-k} \frac{(-1)^i}{i!(k-1)!} = 1.$$

It follows that, for the $(n+1)$ case,

$$
\begin{aligned}
&\frac{1}{n!}+\sum_{k=1}^{n-1}\sum_{i=2}^{n+1-k}\frac{(-1)^i}{i!(k-1)!}\\
&=\frac{1}{n!}+\frac{1}{2\times(n-2)!}+\sum_{k=1}^{n-2}\sum_{i=2}^{n+1-k}\frac{(-1)^i}{i!(k-1)!}\\
&=\frac{1}{n!}+\frac{1}{2\times(n-2)!}+\sum_{k=1}^{n-2}\sum_{i=2}^{n-k}\frac{(-1)^i}{i!(k-1)!}+\sum_{k=1}^{n-2}\frac{(-1)^{n+1-k}}{(n+1-k)!(k-1)!}\\
&=\frac{1}{n!}+\frac{1}{2\times(n-2)!}+1-\frac{1}{(n-1)!}+\sum_{k=1}^{n-2}\frac{(-1)^{n+1-k}}{(n+1-k)!(k-1)!}\\
&=\frac{n^2-3n+2}{2\times n!}+1+\sum_{k=1}^{n-2}\frac{(-1)^{n+1-k}}{(n+1-k)!(k-1)!}\\
&=\frac{n^2-3n+2}{2\times n!}+1+\sum_{t=0}^{n-3}\frac{(-1)^{n-t}}{(n-t)!t!}\\
&=\frac{n^2-3n+2}{2\times n!}+1+\frac{1}{n!}\sum_{t=0}^{n-3}(-1)^{n-t}\binom{n}{t}\\
&=\frac{n^2-3n+2}{2\times n!}+1-\frac{1}{n!}(-1)^2\binom{n}{n-2}-\frac{1}{n!}(-1)^1\binom{n}{n-1}\\
&\quad-\frac{1}{n!}(-1)^0\binom{n}{n}+\frac{1}{n!}\sum_{t=0}^{n}(-1)^{n-t}\binom{n}{t}\\
&=\frac{n^2-3n+2}{2\times n!}+1-\frac{1}{n!}\binom{n}{2}+\frac{1}{n!}\binom{n}{1}-\frac{1}{n!}+\frac{1}{n!}(-1+1)^n\\
&=\frac{n^2-3n+2-n(n-1)+2n-2}{2\times n!}+1\\
&=1.
\end{aligned}
$$

By induction, the result

$$E(X)=1$$

holds for all $n\in\mathbb{N}$.